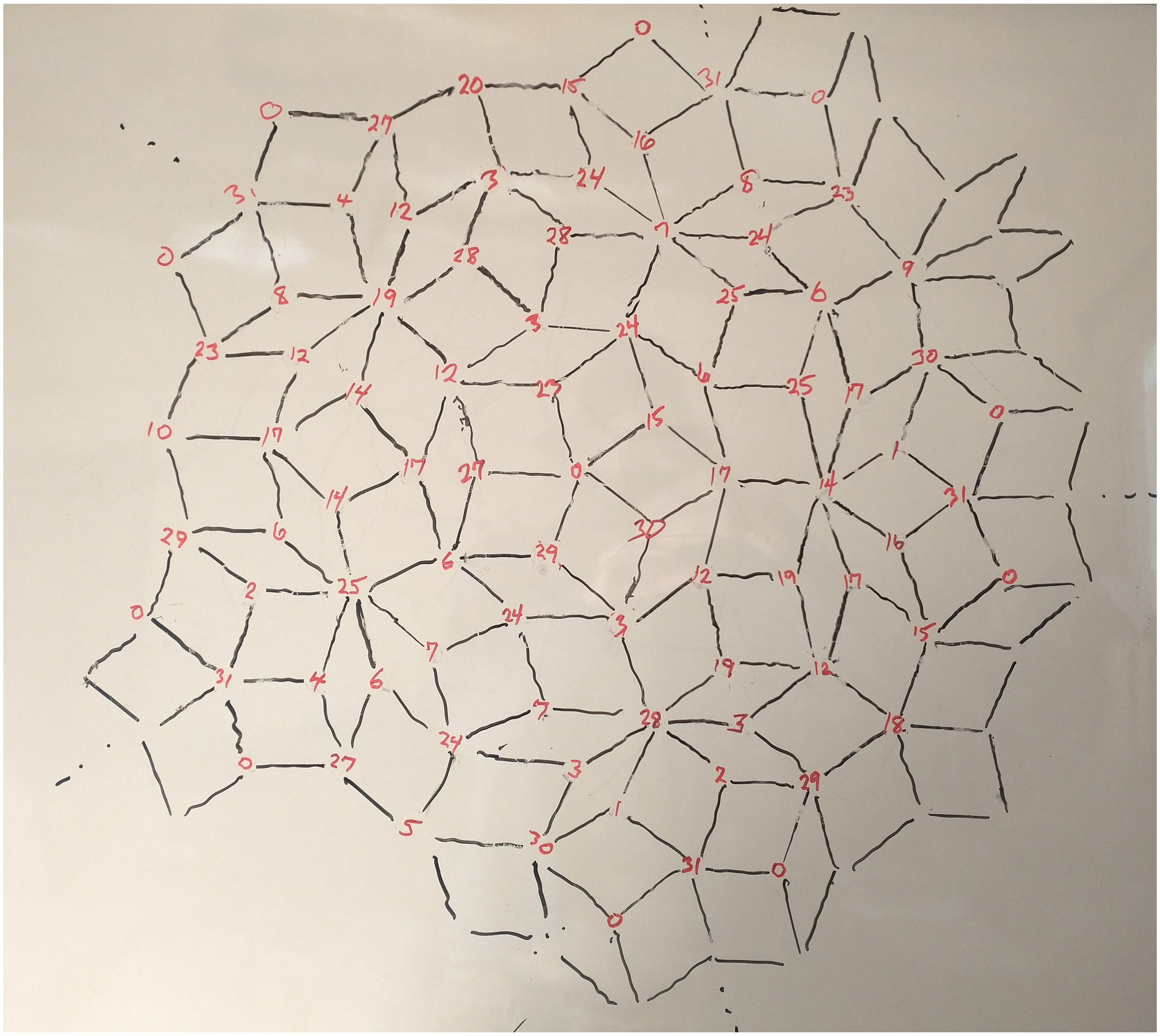
{…}

The magic carpet

It is known[1] that the Penrose tiling is a projection of a section of a five-dimensional cubic lattice onto the plane. In analogy with the cubic lattice in three dimensions, the vertices can be labelled with the integers 0, …, 31 in such a way that the sum around any square is 62, and accordingly the tiling can be so labelled. In part it looks like this:

[1] N. G. de Bruijn, "Algebraic theory of Penrose's non-periodic tilings of the plane [I, II]", *Indagationes Mathematicae* **84**, 38–52, 53–66 (1981).


0
20 15 31
0 27 16 0
3 24 8 23
31 4 12
28 7 24
0 28
8 19 25 6 9
3 24
23 12 6 25 17 30
12 23
14 15 0
10 17
17 27 0 17 14 1 31
14
29 6 30 16
6 29
2 25 12 19 17 0
0 24 3 15
7
31 4 6 19 12
7
28 3 18
24
0 27 3
2 29
1
5 30
31 0
0

This has the following properties:

— There is a central axis around which the graph has a fivefold symmetry, i.e. rotating it any multiple of 72 degrees carries it into itself.

— Similarly there is a reflection symmetry about the horizontal axis, the top and bottom are mirror images.

(These properties hold only for this particular representation of the tiling, which corresponds to a projection onto a plane passing through the origin in five-space.)

— The center is the vertex labelled with 0.

— The horizontal axis passes through the vertices labelled 27, 0, 17, etc.

— The labels of adjacent vertices, regarded as bit-vectors, are related either by the requirement that they agree in exactly one place, as e.g. from the origin

0 = (0 0 0 0 0) <—> (0 1 1 1 1) = 15, (1 0 1 1 1) = 23, (1 1 0 1 1) = 27, (1 1 1 0 1) = 29, (1 1 1 1 0) = 30

or (when crossing a boundary between adjacent five-cubes) as complements, e.g.

17 = (1 0 0 0 1) <—> 14 = (0 1 1 1 0)
24 = (1 1 0 0 0) <—> 7 = (0 0 1 1 1)

(etc.)

— When the graph is rotated counterclockwise 72 degrees, the bit representation of the label does a cyclic shift right, e.g. around the center decahedron

17 = (1 0 0 0 1) —> 24 = (1 1 0 0 0) —> 12 = (0 1 1 0 0)
—> 6 = (0 0 1 1 0) —> 3 = (0 0 0 1 1) —> 17

— Reflection about the horizontal axis carries a label bitwise into its inversion, thus

[a] The numbers on the axis are bitwise palindromic, e.g. left to right

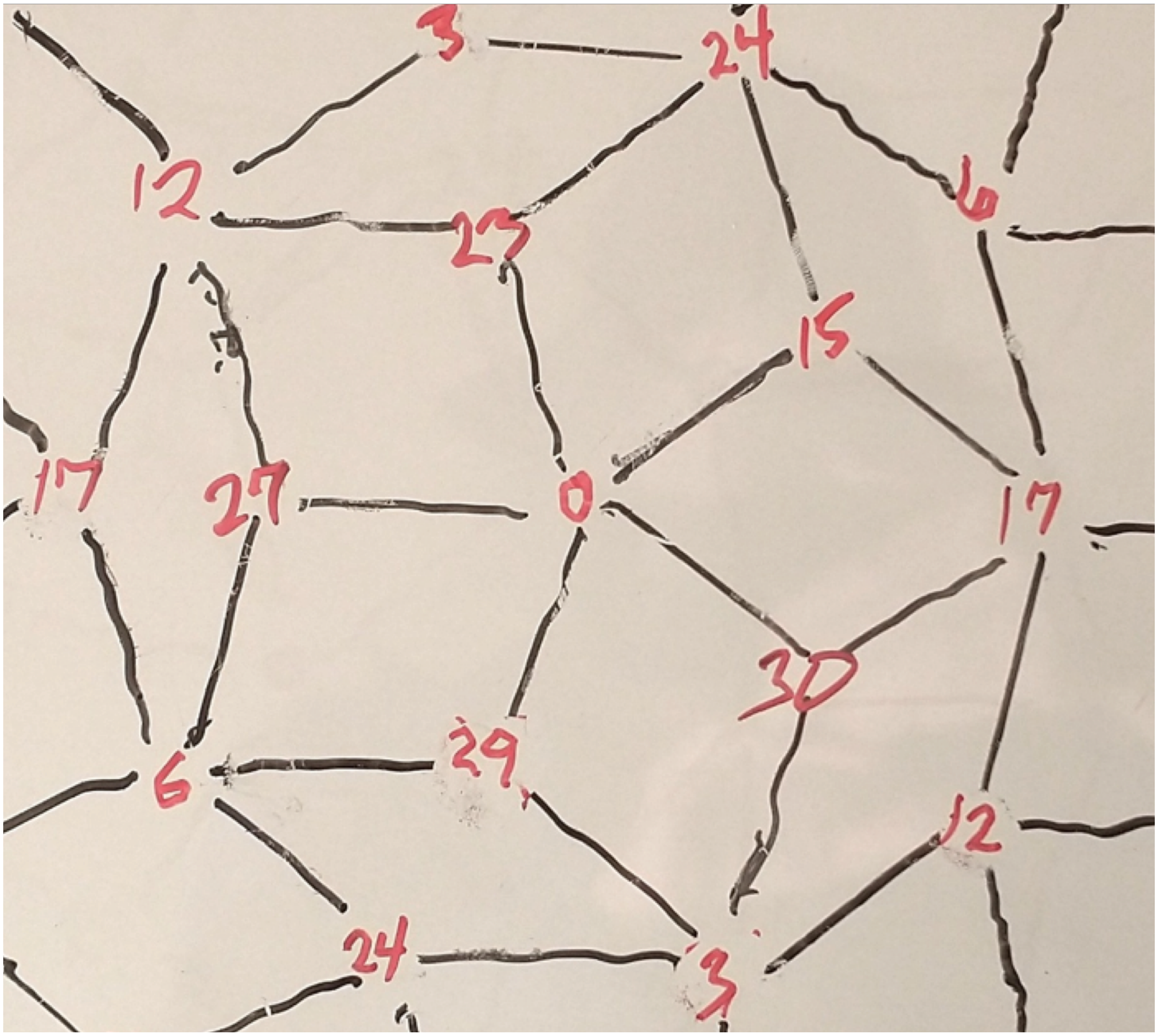
10 = (0 1 0 1 0), 17 = (1 0 0 0 1), 27 =(1 1 0 1 1), 0 = (0 0 0 0 0), 14 = (0 1 1 1 0), 31 = (1 1 1 1 1), and

[b] proceeding right from the center

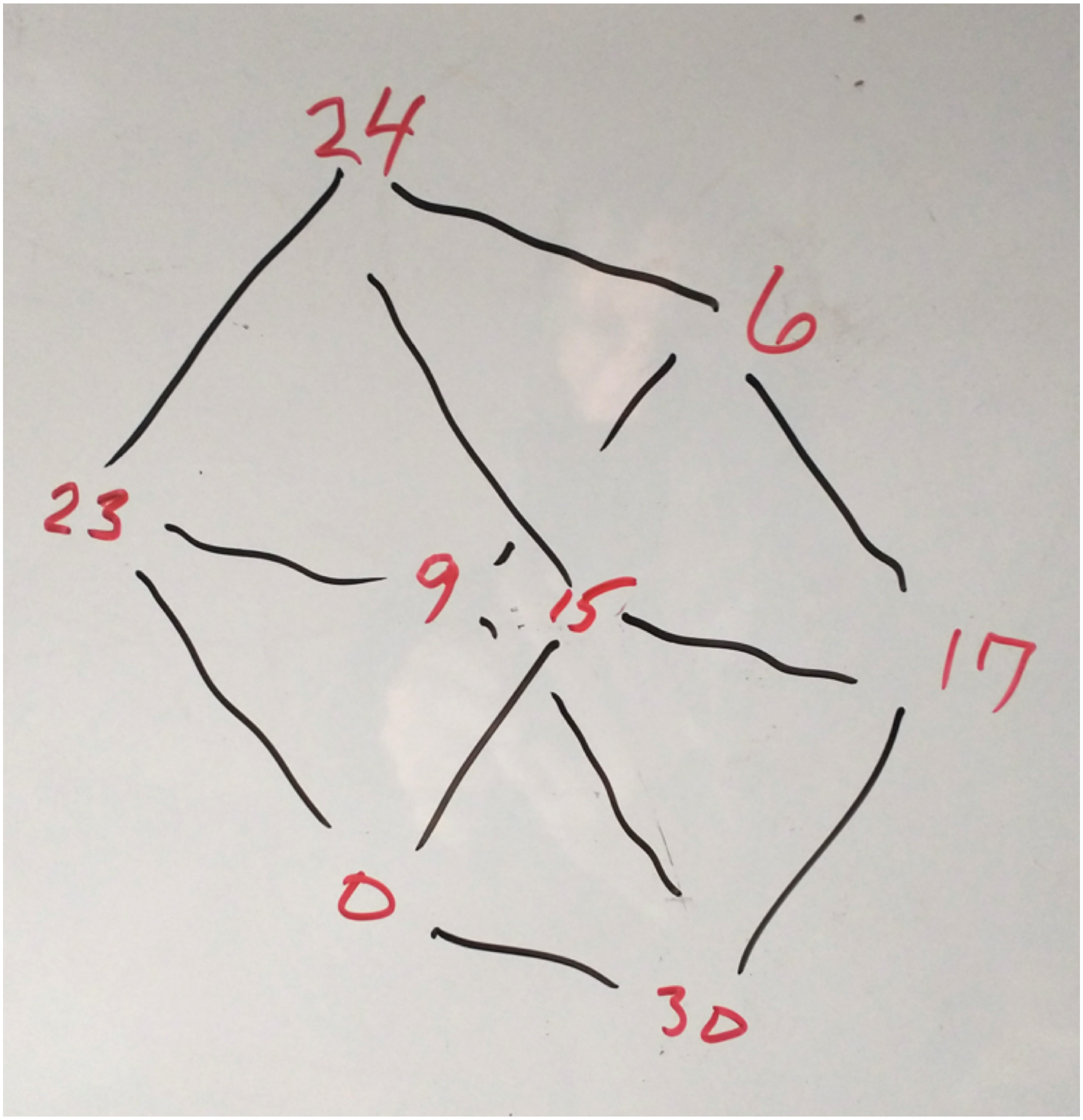
15 = (0 1 1 1 1) <—> 30 = (1 1 1 1 0)
6 = (0 0 1 1 0) <—> 12 = (0 1 1 0 0)
25 = (1 1 0 0 1) <—> 19 = (1 0 0 1 1)

etc.

— Moreover the hidden values of the projected 3-cubes can be inferred as follows: looking, e.g., at the central decahedron



the sum around any square must be 62, and thus the missing values are 9, 20, 10, 5, and 18. (These obey the cyclic shift rule.)


24
6
23
9
15
17
0
30